

# LIVŠIC MEASURABLE RIGIDITY THEOREM FOR $\mathcal{C}^1$ GENERIC VOLUME-PRESERVING ANOSOV SYSTEMS

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**ABSTRACT.** In this paper, we prove that for  $\mathcal{C}^1$  generic volume-preserving Anosov diffeomorphisms of a compact Riemannian manifold, Livšic measurable rigidity theorem holds. We also prove that for  $\mathcal{C}^1$  generic volume-preserving Anosov flows of a compact Riemannian manifold, Livšic measurable rigidity theorem holds.

## 1. INTRODUCTION

Let  $T : M \rightarrow M$  be a diffeomorphism on a compact Riemannian manifold  $M$ . We consider a **cocycle**  $\mathcal{A} : \mathbb{Z} \times M \rightarrow \mathbb{R}$ ; that is, a map satisfying the cocycle relation

$$\mathcal{A}(n_1 + n_2, x) = \mathcal{A}(n_1, T^{n_2}(x)) + \mathcal{A}(n_2, x),$$

for every  $n_1, n_2 \in \mathbb{Z}$  and every  $x \in M$ . Following the definition in cohomological algebra, we call a cocycle  $\mathcal{A}$  a **coboundary** if it satisfies the cohomological equation:

$$(1) \quad \mathcal{A}(n, x) = \Phi(T^n(x)) - \Phi(x),$$

where  $\Phi : M \rightarrow \mathbb{R}$  is a function. Furthermore, two cocycles are called cohomologous if their difference is a coboundary.

It is easy to see that coboundary  $\mathcal{A}$  must have **trivial periodic data**, i.e.

$$(2) \quad \mathcal{A}(n, x) = 0, \quad \forall x \in M, \quad T^n(x) = x.$$

One has three natural questions to propose.

1. Is the necessary condition, trivial periodic data, also a sufficient condition?

2. **Measurable rigidity:** If the cocycle  $\mathcal{A} : \mathbb{Z} \times M \rightarrow \mathbb{R}$  is Hölder continuous, can we get a Hölder continuous solution  $\Phi$  to equation (1) from a measurable solution?

3. **Higher regularity:** If the cocycle  $\mathcal{A} : \mathbb{Z} \times M \rightarrow \mathbb{R}$  is  $\mathcal{C}^r$  for some  $1 \leq r \leq \infty$  or  $r = \omega$ , is a continuous solution to equation (1) also  $\mathcal{C}^r$ ?

Livšic took the lead in considering these three questions for the case when  $f$  is a transitive Anosov diffeomorphism on a compact Riemannian manifold  $M$  [1, 2]. Thus, we call results answering the above questions Livšic theorems. Current research is usually concerned with two variations on this subject, namely altering the base system  $T$  and altering the group  $\mathbb{R}$ . Some of the highlights are [7, 9, 10, 12, 13, 14, 15, 16, 17, 3]. The following theorems are some classical results.

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**Theorem 1.1.** [1, 2, 18, 21] *Let  $T : M \rightarrow M$  be a  $C^1$  transitive Anosov diffeomorphism on a compact Riemannian manifold  $M$  and let  $\phi : M \rightarrow \mathbb{R}$  be a Hölder continuous function.*

- (1) **Existence of solutions.**  $\phi = \Phi(T) - \Phi$  has a Hölder continuous solution  $\Phi$  if and only if  $\sum_{x \in \mathcal{O}} \phi(x) = 0$ , for every  $T$ -periodic orbit  $\mathcal{O}$ .
- (2) **Measurable rigidity.** For any Gibbs measure  $\mu$  with Hölder continuous potential, if there exists a  $\mu$ -measurable solution  $\Phi$  to  $\phi = \Phi(T) - \Phi$ , then there is a continuous solution  $\Psi$ , with  $\Phi = \Psi$ , a.e. $\mu$ .

**Theorem 1.2.** [1, 2, 18, 21] *Let  $\{T^t\}$  be a  $C^1$  transitive Anosov flow on a compact Riemannian manifold  $M$  generated by the vector field  $\xi$  and let  $\phi : M \rightarrow \mathbb{R}$  be a Hölder continuous function.*

- (1) **Existence of solutions.**  $\phi = \Phi'_\xi$  for a Hölder continuous function  $\Phi$  differentiable along the flow if and only if  $\int_0^{t_p} \phi(T^s(p)) ds = 0$  for every periodic point  $p$  with period  $t_p$ .
- (2) **Measurable rigidity.** For any Gibbs measure  $\mu$  with Hölder continuous potential, if there exists a  $\mu$ -measurable solution  $\Phi$  differentiable along the flow such that  $\phi = \Phi'_\xi$   $\mu$ -almost everywhere, then there is a continuous solution  $\Psi$ , with  $\Phi = \Psi$ , a.e. $\mu$ .

In this paper, we are only concerned with measurable rigidity. The proof of measurable rigidity in [18] is based on the Markov partitions and Livšic type theorems for cocycles over shifts of finite type [16], which depend heavily on the equipment of Gibbs measures. For the definition of Gibbs measures, we refer the reader to a classical and short book [19] by Bowen. For other measures, measurable rigidity may not hold.

In this paper, we consider the measurable rigidity for the special measure, volume measure  $m$ . It is known that for  $C^2$  volume-preserving Anosov diffeomorphisms, the volume measure is a Gibbs measure with the Hölder continuous potential

$$\varphi = -\log \det(DT|E^u).$$

However, the volume measure for  $C^1$  volume-preserving Anosov diffeomorphism may not be a Gibbs measure with Hölder continuous potential.

Under a  $C^1$  generic hypothesis, we have the following result.

**Theorem 1.3.** *There exists a residual subset  $\mathcal{G}$  of  $C^1$  Anosov volume-preserving diffeomorphisms on a compact Riemannian manifold  $M$  such that for any  $T \in \mathcal{G}$  and any Hölder continuous function  $\phi : M \rightarrow \mathbb{R}$ , the following three conditions are equivalent:*

- (1)  $\sum_{x \in \mathcal{O}} \phi(x) = 0$ , for every  $T$ -periodic orbit  $\mathcal{O}$ ,
- (2)  $\phi(x) = \Phi(T(x)) - \Phi(x)$  has a continuous solution,
- (3)  $\phi(x) = \Phi(T(x)) - \Phi(x)$ , a.e. for some measurable function  $\Phi$ .

We also get a parallel result for Anosov flows.

**Theorem 1.4.** *There exists a residual subset  $\mathcal{G}$  of  $C^1$  Anosov volume-preserving flows on a compact Riemannian manifold  $M$  such that for any flow  $\{T^t\} \in \mathcal{G}$  and any Hölder continuous function  $\phi : M \rightarrow \mathbb{R}$ , the following three conditions are equivalent:*

- (1)  $\int_0^{t_p} \phi(T^s(p)) ds = 0$  for every periodic point  $p$  with period  $t_p$ ,

- (2)  $\phi = \Phi'_\xi$  for a Hölder continuous function  $\Phi$  differentiable along the flow,
- (3)  $\phi = \Phi'_\xi$  almost everywhere for a measurable function  $\Phi$  differentiable along the flow.

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## 2. PRELIMINARIES

**2.1. Anosov diffeomorphisms.** Assume  $M$  to be a compact Riemannian manifold. Recall that a diffeomorphism  $T : M \rightarrow M$  is called Anosov if there is a  $T$ -invariant splitting

$$TM = E^s \oplus E^u$$

and constants  $C, \rho < 1$ , such that

$$\forall v \in E^s, \|DT^n v\| \leq C\rho^n \|v\|,$$

$$\forall v \in E^u, \|DT^{-n} v\| \leq C\rho^n \|v\|.$$

Now we formulate the Central Limit Theorem for  $\mathcal{C}^2$  volume-preserving Anosov diffeomorphisms. Its proof involves the construction of Markov partition of Anosov diffeomorphisms and the corresponding statistical property of subshifts of finite type.

**Theorem 2.1** (Central Limit Theorem). [19] *Let  $T$  be a  $\mathcal{C}^2$  Anosov volume-preserving diffeomorphism on compact Riemannian manifold  $M$ . Let  $m$  be the volume measure on  $M$ . Let  $\phi$  be a Hölder continuous function on  $M$  with no measurable solution  $\Phi$  to the equation:*

$$\phi(x) - \int \phi(x) dx = \Phi(T(x)) - \Phi(x).$$

*Then  $\phi$  satisfies the Central Limit Theorem with respect to  $T$ , i.e. there exists a constant  $\sigma > 0$  such that for any  $-\infty < \alpha < +\infty$ ,*

$$\lim_{n \rightarrow +\infty} m \left\{ x \in M : \frac{\sum_{i=0}^{n-1} \phi(T^i(x)) - n\bar{\phi}}{\sigma\sqrt{n}} < \alpha \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-\frac{1}{2}u^2} du,$$

*where  $\bar{\phi} = \int_M \phi(x) dm$ . What's more,  $\sigma^2 = \lim_{n \rightarrow +\infty} \frac{\int (\phi_n)^2 dx}{n}$ , where  $\phi_n = \sum_{i=0}^{n-1} (\phi(T^i(x)) - \bar{\phi})$  and  $\sigma$  is called the variance with respect to  $\phi$ .*

**2.2. Anosov flows.** Let us recall the definition of Anosov flow first. Let  $M$  be a compact Riemannian manifold. Let  $T : \mathbb{R} \times M \rightarrow M$  be a  $\mathcal{C}^1$  flow on  $M$  generated by the vector field  $\xi = \frac{d}{dt}(T^t)|_{t=0}$  where  $T^t(\cdot)$  denotes  $T(t, \cdot)$ . Flow  $T : \mathbb{R} \times M \rightarrow M$  is called Anosov if there is a continuous splitting  $TM = E^+ \oplus E^0 \oplus E^-$  with  $E^0$  spanned by  $\xi$  and there are positive constants  $c_1, c_2$  and  $\gamma$  such that

$$\|D(T^t)(\eta)\| \geq c_1 \cdot e^{t\gamma} \cdot \|\eta\|, \forall \eta \in E^+ \text{ and } t \geq 0,$$

$$\|D(T^t)(\eta)\| \leq c_2 \cdot e^{-t\gamma} \cdot \|\eta\|, \forall \eta \in E^- \text{ and } t \geq 0.$$

We use the notation  $\{T^t\}$  as the flow  $T(t, \cdot)$  in this paper. We give the  $\mathcal{C}^r$  topology of flows in the following definition.

**Definition 2.2** ( $C^r$  Topology for Flows). *Let  $\mathcal{F}^r(M)$  be the space of  $C^r$ -flows on  $M$ . Every flow  $\{T^t\} \in \mathcal{F}^r(M)$  restricts to a  $C^r$  map  $T^{[t_0]} : [0, t_0] \times M \rightarrow M$ . Since  $[0, t_0] \times M$  is compact, we may take the usual  $C^r$  topology on  $C^r$  maps  $[0, t_0] \times M \mapsto M$ , and thereby define a  $C^r$  topology on  $\mathcal{F}^r(M)$ . Using the one parameter group property of flows, it is easy to see that the  $C^r$  topology we have defined on  $\mathcal{F}^r(M)$  is independent of  $t_0 > 0$ .*

In the interest of proving Theorem 1.4, we will also use the Central Limit Theorem for  $C^2$  volume-preserving Anosov flows, which is proved by taking advantage of the Markov partition for Anosov flows.

**Theorem 2.3** (Central Limit Theorem for Anosov flows). [20] *Let  $\{T^t\}$  be a  $C^2$  Anosov volume-preserving flow on a compact Riemannian manifold  $M$  generated by vector field  $\xi$  and let  $\phi : M \rightarrow \mathbb{R}$  be a Hölder continuous function on  $M$ . Let  $m$  be the volume measure on  $M$ . If there is no measurable function  $\Phi : M \rightarrow \mathbb{R}$  differentiable along the flow  $\{T^t\}$  such that*

$$\phi = \Phi'_\xi, \text{ a.e.}$$

*then  $\phi$  satisfies the Central Limit Theorem relative to  $\{T^t\}$ , i.e. there exists a constant  $\sigma > 0$  such that for any  $-\infty < \alpha < +\infty$*

$$\lim_{t \rightarrow +\infty} m \left\{ x \in M : \frac{\int_0^t (\phi(T^s(x)) - \bar{\phi}) ds}{\sigma \sqrt{t}} < \alpha \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-\frac{1}{2}u^2} du$$

where  $\bar{\phi} = \int_M \phi(x) dm$ . Moreover,

$$\sigma^2 = \lim_{t \rightarrow +\infty} \frac{\int_M (\phi_t(x))^2 dm}{t}$$

where  $\phi_t$  denotes  $\int_0^t (\phi(T^s(x)) - \bar{\phi}) ds$  and  $\sigma$  is called the variance to function  $\phi$ .

We formulate the definition of cocycles for Anosov flows, introducing some useful terminology along the way. In particular, let  $T : \mathbb{R} \times M \rightarrow M$  be a  $C^1$  Anosov flow, a continuous map  $\mathcal{A} : \mathbb{R} \times M \rightarrow \mathbb{R}$  is a **cocycle** over  $T$  if

$$\mathcal{A}(t_1 + t_2, x) = \mathcal{A}(t_1, T^{t_2}(x)) + \mathcal{A}(t_2, x)$$

for every  $t_1, t_2 \in \mathbb{R}$  and every  $x \in M$ . Cocycle  $\mathcal{A} : \mathbb{R} \times M \rightarrow \mathbb{R}$  is a **coboundary** if there exists a map  $\Phi : M \rightarrow \mathbb{R}$  such that

$$(3) \quad \mathcal{A}(t, x) = \Phi(T^t(x)) - \Phi(x).$$

From now on, we only consider cocycles which are differentiable along the flow. Namely,  $\mathcal{A}(t, p)$  is a  $C^1$  function of  $t$  for all  $p \in M$ . A cocycle  $\mathcal{A} : \mathbb{R} \times M \rightarrow \mathbb{R}$  is called Hölder continuous of exponent  $\alpha \in (0, 1)$  if the map

$$x \mapsto \lim_{t \rightarrow 0} \frac{1}{t} \mathcal{A}(t, x)$$

is **Hölder continuous** of exponent  $\alpha$ . Thus not only the cocycle  $\mathcal{A}$  is differentiable along the flow, but also the derivative of  $\mathcal{A}$  along the flow are Hölder continuous functions on the manifold  $M$ .

There is a natural bijection between cocycles and functions on  $M$ . A cocycle  $\mathcal{A}(t, x)$  is said to be based on a function  $\phi : M \rightarrow \mathbb{R}$  if

$$\mathcal{A}(t, x) = \int_0^t \phi(T^s(x)) ds.$$

The function  $\phi : M \rightarrow \mathbb{R}$  is **the infinitesimal generator**  $\xi(\mathcal{A})$  of cocycle  $\mathcal{A}$ . The existence of this generator is due to the differentiability of the cocycle  $\mathcal{A}$  along the flow. The cocycle  $\mathcal{A}$  is Hölder continuous with exponent  $\alpha$  if and only if the function  $\phi : M \rightarrow \mathbb{R}$  is a Hölder continuous function with exponent  $\alpha$ . Moreover, if the equation

$$\mathcal{A}(t, x) = \Phi(T^t(x)) - \Phi(x).$$

holds, then  $\phi = \Phi'_\xi := d\Phi(\xi)$ .

### 3. PROOF OF THEOREM 1.3.

We now begin the proof of Theorem 1.3. First we state an essential definition.

**Definition 3.1.** *Let  $T$  be a  $\mathcal{C}^1$  Anosov volume-preserving diffeomorphism on a compact Riemannian manifold  $M$ . For any given constants  $C > 0, \tilde{C} > 0, \varepsilon > 0$  and any given periodic point  $p \in M$  with period  $P(p)$ , set*

$$\mathcal{F}_T(\tilde{C}, \varepsilon, p) = \left\{ \phi \mid \phi \text{ is an } \alpha\text{-Hölder continuous function on } M, \int_M \phi dx = 0, \right. \\ \left. \|\phi\|_\alpha \leq \tilde{C}, \sum_{i=0}^{P(p)-1} \phi(T^i(p)) \geq \varepsilon \right\}.$$

We say  $T$  is of  $(C, \tilde{C}, \varepsilon, p)$ -**type**, if there exists a common time  $N$ , such that for any  $\phi \in \mathcal{F}_T(\tilde{C}, \varepsilon, p)$ , there exists at least one moment  $1 \leq k \leq N$  such that,

$$(4) \quad \mu\{x \in M : \phi_k(x) > C\} > \frac{1}{2} - \varepsilon,$$

where  $\phi_k(x) = \sum_{i=0}^{k-1} \phi(T^i(x))$ .

In the following proposition, we use the Central Limit Theorem 2.1 to prove that for any  $(C, \tilde{C}, \varepsilon, p)$ ,  $\mathcal{C}^2$  Anosov volume-preserving diffeomorphisms are  $(C, \tilde{C}, \varepsilon, p)$ -type.

**Proposition 3.2.** *Let  $T$  be a  $\mathcal{C}^2$  Anosov volume-preserving diffeomorphism on a compact manifold  $M$ . For any  $(C, \tilde{C}, \varepsilon, p)$ ,  $T$  is  $(C, \tilde{C}, \varepsilon, p)$ -type.*

*Proof.* Fix constants  $(C, \tilde{C}, \varepsilon)$  and a periodic point  $p$  arbitrarily. According to Theorem 2.1, for any  $\phi \in \mathcal{F}_T(\tilde{C}, \varepsilon, p)$ , there exists  $\sigma > 0$ , such that for any  $\alpha_0 > 0$ , there exists  $N_0 \in \mathbb{N}$  satisfying for any  $n \geq N_0$ ,

$$\begin{aligned} m \left\{ x \in M : \frac{\sum_{i=0}^{n-1} \phi(T^i(x))}{\sigma\sqrt{n}} > \alpha_0 \right\} &\geq \frac{1}{\sqrt{2\pi}} \int_{\alpha_0}^{+\infty} e^{-\frac{1}{2}u^2} du - \frac{\varepsilon}{2} \\ &\geq \frac{1}{2} e^{-\frac{1}{2}\alpha_0^2} - \frac{\varepsilon}{2} \end{aligned}$$

Choose  $\alpha_0$  small enough such that  $\frac{1}{2}e^{-\frac{1}{\sqrt{2}}\alpha_0} \geq \frac{1}{2} - \frac{\varepsilon}{2}$ . Assume  $N_1$  to be an integer satisfying  $\frac{C}{\sigma\sqrt{N_1}} \leq \alpha_0$ . Let  $N(\phi) := \max\{N_0, N_1\}$ . Then, for any  $n \geq N(\phi)$ ,

$$\begin{aligned} m\left\{x \in M : \sum_{i=0}^{n-1} \phi(T^i(x)) > C\right\} &\geq m\left\{x \in M : \frac{\sum_{i=0}^{n-1} \phi(T^i(x))}{\sigma\sqrt{n}} > \alpha_0\right\} \\ &> \frac{1}{2}e^{-\frac{1}{\sqrt{2}}\alpha_0} - \frac{\varepsilon}{2} \\ &> \frac{1}{2} - \varepsilon. \end{aligned}$$

For this fixed time  $N(\phi)$ , there exists a small neighborhood  $\mathcal{U}(\phi)$  of  $\phi$  such that for any function  $\tilde{\phi} \in \mathcal{U}(\phi)$ , we have

$$(5) \quad m\left\{x \in M : \tilde{\phi}_{N(\phi)}(x) > C\right\} > \frac{1}{2} - \varepsilon,$$

where  $\tilde{\phi}_{N(\phi)}(x) = \sum_{i=0}^{N(\phi)-1} \tilde{\phi}(T^i(x))$ .

Due to the compactness of the set  $\mathcal{F}(\tilde{C}, \varepsilon, p)$ , there exists a finite cover  $\mathcal{P} = \{\mathcal{U}(\phi_i)\}_{i=0}^K$  of  $\mathcal{F}(\tilde{C}, \varepsilon, p)$  and thus a common time

$$N = \max_{1 \leq i \leq K} N(\phi_i).$$

This common time  $N$  satisfies the condition we want.  $\square$

Now we prove that  $(C, \tilde{C}, \varepsilon, p)$ -type implies Livšic measurable rigidity.

**Proposition 3.3.** *Let  $T$  be a  $\mathcal{C}^1$  Anosov volume-preserving diffeomorphism on a compact Riemannian manifold  $M$ . Assume that for any  $C > 0$ ,  $\tilde{C} > 0$ ,  $\varepsilon > 0$  and any periodic point  $p$ ,  $T$  is  $(C, \tilde{C}, \varepsilon, p)$ -type. Then for any  $\alpha$ -Hölder continuous function  $\phi : M \rightarrow \mathbb{R}$ , we have three equivalent properties as follows:*

- (1)  $\phi(x) = \Phi(T(x)) - \Phi(x)$  has a continuous solution;
- (2)  $\sum_{x \in \mathcal{O}} \phi(x) = 0$ , for every  $T$ -periodic orbit  $\mathcal{O}$ ;
- (3)  $\phi(x) = \Phi(T(x)) - \Phi(x)$ , a.e. for some measurable function  $\Phi$ .

*Proof.* By Theorem 1.1 and the fact that  $\mathcal{C}^1$  volume-preserving Anosov diffeomorphisms are transitive, we only need to check measurable rigidity, i.e. proving (1) from (3). Assume  $\Phi$  is a measurable solution to  $\phi(x) = \Phi(T(x)) - \Phi(x)$ , a.e., then  $\Phi$  is finite almost everywhere. For any small number  $\varepsilon > 0$ , there exists a constant  $C_\varepsilon > 0$  such that

$$m\{x \in M : \Phi(x) \leq C_\varepsilon\} > 1 - \frac{\varepsilon}{2}.$$

Thus, by the identity  $\phi_n(x) = \Phi(T^n(x)) - \Phi(x)$ , it follows that

$$m\{x \in M : \phi_n(x) \leq 2C_\varepsilon\} > 1 - \frac{\varepsilon}{2}, \quad \forall n \geq 1.$$

If there is no continuous solution for  $\phi(x) = \Phi(T(x)) - \Phi(x)$ , then there must exist a periodic point  $p$  and  $\varepsilon > 0$  such that  $\sum_{i=0}^{P(p)-1} \phi(T^i(p)) \geq \varepsilon$ . However, as  $T$  is  $(C, \tilde{C}, \varepsilon, p)$ -type, for  $C > 2C_\varepsilon$  and function  $\phi$ , there exists a time  $1 \leq k \leq N$ , such that

$$\begin{aligned} (1 - \frac{\varepsilon}{2}) + (\frac{1}{2} - \varepsilon) &\leq m\{x \in M : \phi_k(x) \leq 2C_\varepsilon\} + m\{x \in M : \phi_k(x) \geq C\} \\ &\leq 1, \end{aligned}$$

which is a contradiction. So there exists continuous solution  $\tilde{\Phi} : M \rightarrow \mathbb{R}$  such that  $\phi_n(x) = \tilde{\Phi}(T^n(x)) - \tilde{\Phi}(x)$  and moreover  $\Phi = \tilde{\Phi}$ , a.e..  $\square$

From Proposition 3.2, we have  $\mathcal{C}^2$  Anosov volume-preserving diffeomorphisms are good type. According to A. Avila's result [22] about the regularization of volume-preserving maps:

**Theorem 3.4.** [22] *Smooth maps are  $\mathcal{C}^1$  dense in  $\mathcal{C}^1$  volume-preserving maps.*

By Theorem 3.4 and the  $\mathcal{C}^1$  stability of Anosov systems, we can get  $\mathcal{C}^2$  Anosov volume-preserving diffeomorphisms are dense in  $\mathcal{C}^1$  Anosov volume-preserving diffeomorphisms, which is a key point in our proof.

**Theorem 3.5.** *There exists a residual subset  $\mathcal{G}$  of  $\mathcal{C}^1$  Anosov volume-preserving diffeomorphisms on a compact Riemannian manifold  $M$  such that for any  $T \in \mathcal{G}$  and any  $\phi : M \rightarrow \mathbb{R}$  Hölder continuous, the Livšić theorem holds, i.e. the following three conditions are equivalent:*

- (1)  $\phi(x) = \Phi(T(x)) - \Phi(x)$  has a continuous solution  $\Phi$ ;
- (2)  $\sum_{x \in \mathcal{O}} \phi(x) = 0$ , for every  $T$ -periodic orbit  $\mathcal{O}$ ;
- (3)  $\phi(x) = \Phi(T(x)) - \Phi(x)$ , a.e. for some measurable function  $\Phi$ .

*Proof.* We only need to prove (1) from (3) generically. Take a countable basis  $\mathcal{V} = \{\mathcal{V}_1, \mathcal{V}_2, \dots\}$  of  $M$ . Let  $\mathcal{A}_m^1$  be the set of  $\mathcal{C}^1$  Anosov volume-preserving diffeomorphisms and let  $\mathcal{A}_m^2$  be the set of  $\mathcal{C}^2$  Anosov volume-preserving diffeomorphisms. Denote

$$H_k = \{G \in \mathcal{A}_m^1 \mid \text{there exists a neighborhood } U(G) \subset \mathcal{A}_m^1 \text{ such that for any } G_1 \in U(G), \text{ for any } C > 0, \tilde{C} > 0, \varepsilon > 0 \text{ and for any periodic point of } G_1, p \in \mathcal{V}_k \in \mathcal{V} \text{ with period } P(p) \leq k, G_1 \text{ is } (C, \tilde{C}, \varepsilon, p) - \text{type}\}.$$

It is easy to see that  $H_k$  is open. Set

$$\mathcal{G} := \bigcap_{k \in \mathbb{N}} H_k.$$

Now we prove  $\mathcal{G}$  is the generic set we want.

Let  $\mathcal{A}_m^2$  be the set of  $\mathcal{C}^2$  Anosov volume-preserving diffeomorphisms. In order to proof the density of  $H_k$ , we prove the following lemma first.

**Lemma 3.6.** *The set  $\mathcal{A}_m^2$  is contained in  $H_k$ , for all  $k \geq 1$ .*

*Proof.* Fix any  $T \in \mathcal{A}_m^2$  and  $\mathcal{V}_k$ . We finish our proof by choosing smaller and smaller neighborhoods of  $T$ . By Proposition 3.2, for any  $T \in \mathcal{A}_m^2$  and any  $(C, \tilde{C}, \varepsilon, p)$ ,  $T$  is  $(C, \tilde{C}, \varepsilon, p)$ -type. Thus, there exists  $N$  such that : for any  $\phi \in \mathcal{F}_T(\tilde{C}, \varepsilon, p)$ , there exists  $1 \leq i \leq N$  such that,

$$m\{x \in M : \phi_i(x) > C\} > \frac{1}{2} - \varepsilon.$$

Consider the set

$$\mathcal{F}_G(\tilde{C}, \varepsilon, p) = \left\{ \phi \mid \phi \text{ is an } \alpha\text{-Hölder continuous function on } M, \int_M \phi dx = 0, \|\phi\|_\alpha \leq \tilde{C}, \sum_{i=0}^{P(p)-1} \phi(G^i(p)) \geq \varepsilon \right\},$$

where  $G$  is a  $\mathcal{C}^1$  Anosov volume-preserving diffeomorphism  $\mathcal{C}^1$ -close to  $T$  and  $p \in \mathcal{V}_k$  is a periodic point with period  $P(p) \leq k$  for  $G$ .

There exists a small neighborhood  $W(T)$  of  $T$  such that for any  $G \in W(T)$ ,  $\mathcal{F}_G(\tilde{C}, \varepsilon, p) \subset \mathcal{F}_T(\tilde{C}, \frac{\varepsilon}{2}, q)$ , where  $q$  is the continuation of  $p$  given by structure stability with the same period  $P(p) \leq k$ . Thus, there exists  $N$  such that for any  $\phi \in \mathcal{F}_G(\tilde{C}, \varepsilon, p) \subset \mathcal{F}_T(\tilde{C}, \frac{\varepsilon}{2}, q)$ , we have a time  $1 \leq i \leq N$  such that,

$$m\{x \in M : \phi_{i,T}(x) > C\} > \frac{1}{2} - \frac{\varepsilon}{2},$$

where  $\phi_{i,T} = \sum_{j=0}^{i-1} \phi(T^j(x))$ . Next, there exists a smaller neighborhood  $V(T) \subset W(T)$  of  $T$  such that for any  $G \in V(T)$ , there exists  $N$  such that for any  $\phi \in \mathcal{F}_G(\tilde{C}, \varepsilon, p) \subset \mathcal{F}_T(\tilde{C}, \frac{\varepsilon}{2}, q)$ , we have there exists  $1 \leq i \leq N$  such that,

$$m\left\{x \in M : \phi_{i,G}(x) > \frac{C}{2}\right\} > \frac{1}{2} - \frac{\varepsilon}{2}.$$

Taking the uniform hyperbolicity of  $T$  into account, there are only finite  $p \in \mathcal{V}_k$  with period  $P(p) \leq k$  for every  $G \in V(T)$ . Thus we get another smaller neighborhood  $U(T) \subset V(T)$ , such that for any  $G \in U(T)$ , and any constants  $C > 0, \tilde{C} > 0, \varepsilon > 0$  and any periodic point  $p \in \mathcal{V}_k$  with period  $P(p) \leq k$ ,  $G$  is  $(C, \tilde{C}, \varepsilon, p)$ -type.

Thus,  $T \in H_k$ . This completes the proof of this lemma.  $\square$

So Lemma 3.6 implies that  $H_k$  is  $\mathcal{C}^1$  dense in  $\mathcal{C}^1$  Anosov volume-preserving diffeomorphisms and then we get that  $\mathcal{G}$  is a generic set.

It is easy to see from the definition that for every diffeomorphism  $G \in \mathcal{G}$  and tuple  $(C, \tilde{C}, \varepsilon, p)$ ,  $G$  is  $(C, \tilde{C}, \varepsilon, p)$ -type. By Proposition 3.3, we finish the proof.  $\square$

#### 4. PROOF OF THEOREM 1.4

The argument for Anosov flows proceeds in an almost identical fashion as in the previous section, *mutatis mutandis*. Theorem 1.2 and Theorem 2.3, instead of Theorem 1.1 and Theorem 2.1, are needed in the proof of Theorem 1.4.

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